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Exact Solution of Mixed Problems for Variable Coefficient One-Dimensional Diffusion Equation

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Abstract—This paper deals with diffusion problems modeled by the equation $a(t)u_{xx} = u_t$, $x > 0$, $t > 0$, $u(x, 0) = c(x)$ together with the boundary condition $u(0, t) = b(t)$ or $u_x(0, t) = b(t)$. By using Fourier transforms, existence conditions and exact solutions of the above mixed problems are given.

Keywords—Diffusion equation, Variable coefficient, Fourier transform, Explicit solution.

1. INTRODUCTION

In the evaluation of microwave heating processes, the constant coefficient model often leads to misleading results due to the complexity of the field distribution within the oven and the variation of the dielectric properties of the material with several parameters [1,2]. Constant coefficient models are inadequate in drying processes where the humidity changes are important, or in propagation problems in ferrite materials [1]. Linear flow of heat in the semi-infinite solid where thermal properties are time dependent but independent of position are modeled by the one-dimensional diffusion equation

$$a(t)\, u_{xx}(x,t) = u_t(x,t), \qquad x > 0, \quad t > 0, \tag{1}$$

where $a(t)$ is the thermometric conductivity [3]. Economic problems related to the valuation of options with time-varying interest rate and volatility can be modeled in terms of (1) starting from Black-Scholes formulae, see [4, p. 101].

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This paper deals with mixed problems described by (1) together with the initial condition

$$u(x,0) = c(x), \qquad x > 0, \tag{2}$$

and one of the two boundary conditions

$$u(0,t) = b(t), \qquad t > 0, \tag{3}$$
$$u_x(0,t) = b(t), \qquad t > 0. \tag{4}$$

It is well known that explicit solutions of problems have undoubtable advantages versus numerical solutions such as the possibility to check the correctness of the model, the study of the variation of the solution according with the data and avoiding accumulation errors. So, in spite of the wide use of numerical methods for the treatment of variable coefficient partial differential equations, it is worthy to obtain explicit solutions of variable coefficient problems. The aim of this paper is twofold. First, to show that the transformation integrals approach can be used to deal with variable coefficient problems, and second, to obtain explicit solutions of initial-boundary value problems related to the one-dimensional time-dependent diffusion equation. The organization of the paper is as follows. Section 2 deals with problem (1)–(3) as well as (1),(2),(4), for the case where the initial condition of both problems is $u(x,0) = 0$, $x > 0$ and exact solutions are found using the sine and cosine Fourier transform. The general case where $c(x)$ in (2) is a nonzero function is treated in Section 3. Throughout this paper, erf (x) denotes the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds,$$

$\mathcal{F}_s$ denotes the sine Fourier transform, and $\mathcal{F}_c$ the cosine Fourier transform. For the sake of clarity in the presentation, we recall the convolution theorem for the cosine Fourier transform: if $\mathcal{F}_c\{f(x)\} = F_c(w)$ and $\mathcal{F}_c\{g(x)\} = G_c(w)$, then

$$\int_0^\infty F_c(w)\, G_c(w) \cos(wx)\, dw = \frac{1}{2} \int_0^\infty f(y) \left[g(x+y) + g(|x-y|) \right] dy, \tag{5}$$

see [5, p. 239]. By [5, p. 240] and with previous notation, one gets

$$\int_0^\infty F_s(w)\, G_c(w) \sin(wx)\, dw = \frac{1}{2} \int_0^\infty f(y) \left[g(|x-y|) - g(x+y) \right] dy. \tag{6}$$

Finally, we recall that if $b \geq 0$, then (see [6, p. 480])

$$\int_0^\infty e^{-x^2} \cos(2bx)\, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}, \tag{7}$$

and differentiation of definite integral with respect to a parameter [6, p. 18]

$$\frac{d}{da} \int_{\psi(a)}^{\varphi(a)} f(x,a)\, dx = f(\varphi(a), a) \frac{d\varphi(a)}{da} - f(\psi(a), a) \frac{d\psi(a)}{da} + \int_{\psi(a)}^{\varphi(a)} \frac{d}{da} f(x,a)\, dx. \tag{8}$$

2. EXPLICIT EXACT SOLUTION FOR THE ZERO INITIAL CONDITION CASE

Let us consider problem (1)–(3) with $c(x) = 0$ for $x > 0$, and assume the hypothesis

$$\text{there exist } \delta > 0 \text{ such that } a(t) \geq \delta \text{ for all } t > 0. \tag{9}$$

Let us consider the unknown u as a function of the variable x for a fixed value of t, and let

$$\mathcal{U}(t)(w) = \mathcal{F}_s \{u(\cdot, t)\}(w). \tag{10}$$

By applying the sine transform to problem (1)–(3) with $c(x) = 0$, and taking into account the properties of the sine Fourier transform, see [7, p. 52], it follows that $\mathcal{U}(t)$ satisfies the problem

$$\frac{d}{dt}\mathcal{U}(t)(w) = -w^2 a(t)\mathcal{U}(t)(w) + wa(t)b(t), \qquad U(0)(w) = 0, \tag{11}$$

where $w > 0$ is a parameter.

Assuming that $b(t)$ is differentiable, solving (11) and integrating by parts, it follows that

$$\begin{aligned}\mathcal{U}(t)(w) &= w \int_0^t e^{-w^2 \int_0^t a(s)\,ds} a(v)b(v)\,dv \\ &= \frac{b(t)}{w} - \frac{b(0)}{w} e^{-w^2 \int_0^t a(s)\,ds} - \frac{1}{w}\int_0^t e^{-w^2 \int_v^t a(s)\,ds} b'(v)\,dv.\end{aligned} \tag{12}$$

By using the inverse Fourier sine transform, taking into account hypothesis (5) and Fubini's Theorem, by (10) and (12), one gets

$$\begin{aligned}u(x,t) &= \frac{2}{\pi}\int_0^t \int_0^\infty \frac{\sin(wx)}{w} e^{-w^2 \int_v^t a(s)\,ds} b'(v)\,dv\,dw \\ &+ b(t) - \frac{2}{\pi}b(0)\int_0^\infty \frac{\sin(wx)}{w} e^{-w^2 \int_0^t a(s)\,ds}\,dw.\end{aligned} \tag{13}$$

Let $J_1(x)$ and $J_2(x)$ be defined by

$$\begin{aligned}J_1(x) &= \int_0^\infty \frac{\sin(wx)}{w} e^{-w^2 \int_0^t a(s)\,ds}\,dw, \\ J_2(x) &= \int_0^\infty \frac{\sin(wx)}{w} e^{-w^2 \int_v^t a(s)\,ds}\,dw.\end{aligned} \tag{14}$$

By the Leibniz formula for the differentiation of parametric integrals, it follows that

$$\begin{aligned}J_1'(x) &= \int_0^\infty \cos(wx) e^{-w^2 \int_0^t a(s)\,ds}\,dw \\ &= \frac{1}{\sqrt{\int_0^t a(s)\,ds}} \int_0^\infty e^{-z^2} \cos\left(\frac{zx}{\sqrt{\int_0^t a(s)\,ds}}\right) dz,\end{aligned} \tag{15}$$

and by (7), one gets

$$J_1'(x) = \frac{\sqrt{\pi}}{2\sqrt{\int_0^t a(s)\,ds}} e^{-x^2/(4\int_0^t a(s)\,ds)}. \tag{16}$$

Since $J_1(0) = 0$, by (15), it follows that

$$\begin{aligned}J_1(x) &= \int_0^x J_1'(s)\,ds = \frac{\sqrt{\pi}}{2\sqrt{\int_0^t a(s)\,ds}} \int_0^x e^{-(\xi^2/4\int_0^t a(s)\,ds)}\,d\xi \\ &= \sqrt{\pi}\,\mathrm{erf}\left(\frac{x}{2\sqrt{\int_0^t a(s)\,ds}}\right).\end{aligned} \tag{17}$$

By (13) and (16) and using that $J_2(0) = 0$, one gets

$$J_2(x) = \sqrt{\pi}\,\mathrm{erf}\left(\frac{x}{2\sqrt{\int_v^t a(s)\,ds}}\right). \tag{18}$$

By (13),(16)–(18), it follows that

$$u(x,t) = b(t) - b(0)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\int_0^t a(s)\,ds}}\right) - \int_0^t b'(v)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\int_v^t a(s)\,ds}}\right) dv. \tag{19}$$

Let us prove that $u(x,t)$ defined by (19) satisfies condition $u(x,0) = 0$, for $x > 0$ and (3). If $t > 0$ is fixed, from (19), it follows that

$$\lim_{x\to 0+} u(x,t) = \lim_{x\to 0+} \left(b(t) - b(0)\,\mathrm{erf}\,(0+)\right) = b(t).$$

Otherwise, if $x > 0$ is fixed, by (17), it follows that

$$\lim_{x\to 0+} u(x,t) = b(0) - b(0)\,\mathrm{erf}\,(\infty) - \lim_{x\to 0+} \int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{\int_v^t a(s)\,ds}}\right) b'(v)\,dv. \tag{20}$$

By the Mean Value Theorem 7.37 of [8, p. 200], one gets the existence of some ξ with $0 < \xi < t$ such that

$$\int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{\int_v^t a(s)\,ds}}\right) b'(v)\,dv = \mathrm{erf}\,(\infty)\int_\xi^t b'(v)\,dv = \int_\xi^t b'(v)\,dv = b(t) - b(\xi).$$

Hence,

$$\lim_{x\to 0} \int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{\int_v^t a(s)\,ds}}\right) b'(v)\,dv = 0, \tag{21}$$

and by (19) and (20), one concludes that $\lim_{x\to 0} u(x,t) = 0$.

Finally, we prove that $u(x,t)$ defined by (19) satisfies equation (1). Note that by definition of the error function and the chain rule, it follows that

$$\frac{d}{dx}\left(\mathrm{erf}\left(\frac{x}{2\sqrt{\int_v^t a(s)\,ds}}\right)\right) = \frac{1}{\sqrt{\pi\int_v^t a(s)\,ds}}\exp\left(-\frac{x^2}{4\int_v^t a(s)\,ds}\right). \tag{22}$$

By (8), (18), (21), and (22), it is easy to check that

$$\begin{aligned}
a(t)u_{xx}(x,t) &= \frac{b(0)xa(t)}{2\sqrt{\pi}}\left(\int_0^t a(s)\,ds\right)^{-3/2}\exp\left(-\frac{x^2}{4\int_0^t a(s)\,ds}\right)\\
&\quad + \frac{xa(t)}{2}\int_0^t \exp\left(-\frac{x^2}{4\int_v^t a(s)\,ds}\right)\left(\int_v^t a(s)\,ds\right)^{-3/2} b'(v)\,dv\\
&= a(t)u_t(x,t), \qquad x > 0, \quad t > 0.
\end{aligned}$$

Let us consider the problem described by (1), (2), and (4) with $c(x) = 0$ and let us denote

$$\mathcal{U}(t)(w) = \mathcal{F}_c\left\{u(\cdot,t)\right\}(w). \tag{23}$$

Taking into account the properties of the Fourier cosine transform [7, p. 52] and (4) it follows that

$$\mathcal{F}_c\{u_{xx}\}(w) = -u_x(0,t) - w^2\mathcal{U}(t)(w) = -b(t) - w^2\mathcal{U}(t)(w). \tag{24}$$

By applying the Fourier cosine transform to problem (1),(2),(4) with $c(x) = 0$, and by (23) and (24), one gets the transformed initial value problem

$$\frac{d\mathcal{U}(t)(w)}{d} = -a(t)b(t) - w^2a(t)\mathcal{U}(t)(w), \qquad \mathcal{U}(0)(w) = 0, \tag{25}$$

whose solution is

$$\mathcal{U}(t)(w) = -\int_0^t e^{-w^2\int_v^t a(s)\,ds}a(v)b(v)\,dv. \tag{26}$$

By applying the inverse Fourier cosine transform to (26), by (23), Fubini's Theorem, and (9), it follows that

$$\begin{aligned} u(x,t) &= \frac{2}{\pi}\int_0^\infty \mathcal{U}(t)(w)\cos(wx)\,dw \\ &= -\frac{2}{\pi}\int_0^t \left(\int_0^\infty e^{-w^2\int_v^t a(s)\,ds}\cos(wx)\,dw\right)a(v)b(v)\,dv. \end{aligned} \tag{27}$$

Using (7) and (27), one gets

$$\begin{aligned} \int_0^\infty e^{-w^2\int_v^t a(s)\,ds}\cos(wx)\,dw &= \frac{\sqrt{\pi}}{2\sqrt{\int_v^t a(s)\,ds}}e^{-(x^2/4\int_v^t a(s)\,ds)}, \\ u(x,t) &= -\frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-(x^2/4\int_v^t a(s)\,ds)}}{\sqrt{\int_v^t a(s)\,ds}}a(v)b(v)\,dv. \end{aligned} \tag{28}$$

Summarizing, the following result has been established.

THEOREM 1. *Let $a(t)$ be a continuous positive function satisfying condition (9) and let $c(x) = 0$ for all $x > 0$.*

(i) *If $b(t)$ is a continuously differentiable function, then a solution of problem (1)–(3) is given by (18).*
(ii) *If $b(t)$ is continuous, then a solution of problem (1),(2),(4) is given by (28).*

REMARK 2. Solutions proposed by Theorem 1 coincide with those given in [7, pp. 54,56] for the case where $a(t) = a$ for all $t > 0$.

3. THE GENERAL CASE

Throughout this section, we assume that

$$c(x) \text{ is integrable and absolutely integrable in } [0,\infty[. \tag{29}$$

We begin this section dealing with problem (1)–(3) or (1),(2),(4) with the condition $b(t) = 0$ for all $t > 0$. In accordance with (10), we regard the unknown u as a function of the active variable x, $u = u(\cdot,t)$ for a fixed parameter $t > 0$. Let us address problem (1)–(3) by applying the Fourier sine transform. Taking into account the properties of the Fourier sine transform, problem (1)–(3) is transformed into

$$\frac{d\mathcal{U}(t)(w)}{d} = -a(t)w^2\mathcal{U}(t)(w), \qquad \mathcal{U}(0)(w) = \mathcal{C}(w), \tag{30}$$

where

$$\mathcal{C}(w) = \mathcal{F}_s\{c(x)\}(w). \tag{31}$$

The solution of (30) takes the form

$$\mathcal{U}(t)(w) = \mathcal{C}(w)e^{-w^2 \int_v^t a(s)\, ds}. \tag{32}$$

It is easy to check by direct computation or using the properties of the Fourier sine transform that under hypothesis (9), the function

$$g_t(x) = \frac{1}{\sqrt{\pi \int_0^t a(s)\, ds}} e^{-x^2/(\int_0^t a(s)\, ds)} \tag{33}$$

is integrable in the positive real line and

$$\mathcal{F}_c\{g_t(x)\}(w) = e^{-w^2 \int_0^t a(s)\, ds}. \tag{34}$$

Thus, (32) takes the form

$$\mathcal{F}_s\{u(\cdot,t)\}(w) = \mathcal{F}_s\{c(x)\}(w)\mathcal{F}_c\{g_t(x)\}(w), \tag{35}$$

and by (6) and (35), it follows that

$$u(x,t) = \frac{1}{2\sqrt{\pi \int_0^t a(s)\, ds}} \int_0^\infty c(y)\left\{e^{-((x-y)^2/4\int_0^t a(s)\, ds)} - e^{-((x+y)^2/4\int_0^t a(s)\, ds)}\right\} dy. \tag{36}$$

By Theorem 7.3 of [5, p. 208], by the continuity of $c(x)$, it follows that

$$\lim_{t\to 0+} u(x,t) = c(x), \qquad \text{for a fixed value of } x > 0.$$

By (36), one gets

$$\lim_{x\to 0+} u(x,t) = 0, \qquad \text{for a fixed value of } t > 0.$$

Finally, by the Leibniz rule for the differentiation of parametric integrals and (36), it follows that

$$\begin{aligned}
u_t(x,t) &= -\frac{a(t)}{2\sqrt{\pi}\left(\int_0^t a(s)\, ds\right)^{3/2}} \int_0^\infty c(y)\left\{e^{-((x-y)^2/4\int_0^t a(s)\, ds)} - e^{-((x+y)^2/4\int_0^t a(s)\, ds)}\right\} dy \\
&+ \frac{a(t)}{2\sqrt{\pi \int_0^t a(s)\, ds}} \int_0^\infty c(y)\left\{\frac{(x-y)^2}{4\int_0^t a(s)\, ds} e^{-((x-y)^2/4\int_0^t a(s)\, ds)} + \frac{(x+y)^2}{4\int_0^t a(s)\, ds} e^{-((x+y)^2/4\int_0^t a(s)\, ds)}\right\} dy \\
&= a(t)\frac{1}{2\sqrt{\pi \int_0^t a(s)\, ds}}\left[\int_0^\infty c(y)\left\{\left(\frac{(x-y)^2}{4\int_0^t a(s)\, ds} - \frac{1}{2\int_0^t a(s)\, ds}\right) e^{-((x-y)^2/4\int_0^t a(s)\, ds)}\right.\right. \\
&\qquad \left.\left. + \left(\frac{1}{2\int_0^t a(s)\, ds} - \frac{(x+y)^2}{4\int_0^t a(s)\, ds}\right) e^{-((x+y)^2/4\int_0^t a(s)\, ds)}\right\}\right] dy \\
&= a(t)u_{xx}(x,t), \qquad x > 0, \quad t > 0.
\end{aligned}$$

Hence, $u(x,t)$ defined by (36) solves problem (1)–(3) with $b(t) = 0$.

Consider now problem (1),(2),(4) with $b(t) = 0$ and notation (23). By applying the Fourier cosine transform and taking into account the properties relating the transform of the derivatives, problem (1),(2),(4) is transformed into

$$\frac{d\mathcal{U}(t)(w)}{d} = -a(t)w^2\mathcal{U}(t)(w), \qquad \mathcal{U}(0)(w) = \mathcal{C}_1(w), \tag{37}$$

where

$$\mathcal{C}_1(w) = \mathcal{F}_c\{c(\cdot)\}(w). \tag{38}$$

By (33),(34), solving (37) and applying the inverse Fourier cosine, it follows that

$$\mathcal{U}(t)(w) = \mathcal{C}_1(w)e^{-w^2\int_0^t a(s)\,ds}, \tag{39}$$

$$\begin{aligned} u(x,t) &= \frac{2}{\pi}\int_0^\infty \mathcal{C}_1(w)e^{-w^2\int_0^t a(s)\,ds}\cos(wx)\,dw \\ &= \frac{2}{\pi}\int_0^\infty \mathcal{F}_c\{c\}(w)\mathcal{F}_c\{g_t\}(w)\cos(wx)\,dw. \end{aligned} \tag{40}$$

By property (5) and (40), one gets

$$u(x,t) = \frac{1}{2\sqrt{\pi\int_0^t a(s)\,ds}}\int_0^\infty c(y)\left\{e^{-((x+y)^2/4\int_0^t a(s)\,ds)} - e^{-((x-y)^2/4\int_0^t a(s)\,ds)}\right\}dy. \tag{41}$$

Summarizing, the following result has been established.

THEOREM 3. *Let $a(t)$ be a positive continuous function satisfying condition (9), let $b(t) = 0$ for all $t > 0$ and assume hypothesis (29).*

(i) *A solution of problem (1)–(3) is given by (36).*
(ii) *A solution of problem (1),(2),(4) is given by (41).*

REMARK 4. If $a(t) = a$ is a constant function, the solutions provided by Theorem 3 coincide with the expressions given in [5, pp. 239,240].

COROLLARY 5. *Let $a(t)$ be a positive continuous function satisfying (9), let $b(t)$ be a continuous function, and assume that $c(x)$ satisfies (29).*

(i) *If $b(t)$ is continuously differentiable in the positive real line, then a solution of problem (1)–(3) is given by*

$$\begin{aligned} u(x,t) = b(t) &- b(0)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\int_0^t a(s)\,ds}}\right) - \int_0^t b'(v)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\int_v^t a(s)\,ds}}\right)dv \\ &+ \frac{1}{2\sqrt{\pi\int_0^t a(s)\,ds}}\int_0^\infty c(y)\left\{e^{-((x-y)^2/4\int_0^t a(s)\,ds)} - e^{-((x+y)^2/4\int_0^t a(s)\,ds)}\right\}dy. \end{aligned} \tag{42}$$

(ii) *A solution of problem (1),(2),(4) is given by*

$$\begin{aligned} u(x,t) = &-\frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-x^2/(4\int_v^t a(s)\,ds)}}{\sqrt{\int_v^t a(s)\,ds}}a(v)b(v)\,dv \\ &+ \frac{1}{2\sqrt{\pi\int_0^t a(s)\,ds}}\int_0^\infty c(y)\left\{e^{-((x+y)^2/4\int_0^t a(s)\,ds)} - e^{-((x-y)^2/4\int_0^t a(s)\,ds)}\right\}dy. \end{aligned} \tag{43}$$

PROOF. Note that if u_1 is a solution of problem

$$\begin{aligned} a(t)(u_1)_{xx}(x,t) &= (u_1)_t(x,t), && x>0, \quad t>0, \\ u_1(x,0) &= 0, && x>0, \\ u_1(0,t) &= b(t), && t>0, \end{aligned}$$

and u_2 is a solution of

$$\begin{aligned} a(t)\,(u_2)_{xx}(x,t) &= (u_2)_t(x,t), && x>0, \quad t>0, \\ u_2(x,0) &= c(x), && x>0, \\ u_2(0,t) &= 0, && t>0, \end{aligned}$$

then $u = u_1 + u_2$ is a solution of problem (1)–(3). In an analogous way, if v_1 is a solution of problem

$$\begin{aligned} a(t)\,(v_1)_{xx}(x,t) &= (v_1)_t(x,t), && x>0, \quad t>0, \\ v_1(x,0) &= 0, && x>0, \\ (v_1)_x(0,t) &= b(t), && t>0, \end{aligned}$$

and v_2 is a solution of problem

$$\begin{aligned} a(t)\,(v_2)_{xx}(x,t) &= (v_2)_t(x,t), && x>0, \quad t>0, \\ v_2(x,0) &= c(x), && x>0, \\ (v_2)_x(0,t) &= 0, && t>0, \end{aligned}$$

then $v = v_1 + v_2$ is a solution of problem (1),(2),(4). Now the result is a direct consequence of Theorems 1 and 3. ∎

REFERENCES

1. A.C. Metaxas and R.J. Meredith, *Industrial Microwave Heating*, Peter Peregrinus, London, (1983).
2. D.M. Pozar, *Microwave Engineering*, Addison-Wesley, New York, (1990).
3. H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids*, Oxford Univ. Press, Oxford, (1995).
4. P. Wilmott, S. Howison and J. Dewynee, *The Mathematics of Financial Derivatives*, Cambridge Univ. Press, Cambridge, (1995).
5. G.B. Folland, *Real Analysis*, John-Wiley, New York, (1984).
6. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Third Edition, Academic Press, New York, (1998).
7. L. Debnath, *Integral Transforms and Their Applications*, CRC Press, New York, (1995).
8. T.M. Apostol, *Mathematical Analysis*, Addison-Wesley, New York, (1997).